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Three-Charge Supertubes in a Rotating Black Hole Background

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ABSTRACT: The low velocity scattering of a D0-F1 supertube in the background of a BMPV black hole has been considered in (hep-th/0505044). Here we extend the analysis to the case of the D0-D4-F1 supertube of (hep-th/0402144). We find that, similarly to the two-charge case, when the supertube moves in the black hole background there can exist a position of stable equilibrium identical to the location of the corresponding static BPS solution. As with the D0-F1 supertube, low velocity mergers with the black hole can violate the BMPV angular momentum bound $J^2 < N_{D0}N_{D4}N_{F1}$, although such processes are always accompanied by a potential barrier. Partial correspondence with the exact supergravity solution of (hep-th/0512157) is established.

KEYWORDS: Black holes in string theory, Supertubes, BMPV.

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1. Introduction

In their original worldvolume formulation, supertubes are solutions of the Dirac-Born-Infeld (DBI) low-energy effective action for D-branes in Type IIA string theory. They take the form of tubular D-brane configurations, possessing static electric and magnetic fields on the D-brane worldvolume that produce a nontrivial amount of angular momentum. They also preserve the same supersymmetries as the three-charge rotating supersymmetric black

hole, called the BMPV black hole after the authors of [16], which makes them ideal probes of such a black hole.

Not being exact supergravity solutions, the worldvolume supertubes do not incorporate backreaction of the supertube on the geometry, and do not account for certain interactions. However, researchers have also found supergravity supertubes, which are explicit solutions of Type IIA and Type IIB supergravity actions [4, 5]. Recent work *e.g.* [6, 7, 8, 9, 10] has shown that supergravity supertubes are part of a larger class of solutions of supergravity that saturate a Bogomol’nyi-Prasad-Sommerfeld (BPS) bound. These supersymmetric solutions in general possess three charges, three dipole moments and two independent angular momenta; they include supertubes, BPS black rings and BMPV black holes, and arbitrary superpositions of these objects. Moreover, certain excited states of two-charge and three-charge supertubes contain event horizons and correspond to (non-supersymmetric) higher dimensional lifts of black rings, sometimes called black tubes [6, 10]. We point out that our use of the term ‘supertube’ always refers to a BPS object without an event horizon.

Supertubes with two charges have one dipole moment and one nonzero angular momentum, and the supergravity and worldvolume formulations of these objects agree completely [6]. In the case of three-charge supertubes, this correspondence is not as clear-cut. As mentioned, generic three-charge supergravity supertubes possess three dipole moments and two angular momenta, even in the absence of other physical influences. In contrast, both of the worldvolume descriptions provided in [1] have two dipole moments and one nonzero angular momentum; that in [6], based on M-theory, has three dipole moments, but again only a single nonzero angular momentum. We will revisit some of these discrepancies in Section 6.

Despite its shortcomings, a great advantage of a D-brane worldvolume description over

a full supergravity solution is that a time-dependent, low velocity scattering calculation can be performed in a straightforward manner. Thus we can consider not only mergers of the supertube with a BMPV black hole that occur in the adiabatic limit, but also scattering behavior that occurs when the supertube is given a slow velocity with respect to the black hole. The present treatment utilizes the D6 brane worldvolume analysis of [1], for ease of comparison with earlier results, particularly those of [2]. Using the DBI action, the authors of [2] undertook an investigation of supertube scattering in the vicinity of a BMPV black hole, using a D2 brane supertube with D0 and F1 charge. Here we extend the analysis to D0-D4-F1 supertubes. It is instructive to determine how the mergers differ in the two-charge and three-charge situations. A further comparison can be made, in the case of adiabatic mergers, between the worldvolume and the supergravity schemes.

We find that similarly to the two-charge case, adiabatic mergers do not take place when the circumferential angular momentum j_1 exceeds a certain critical value j_{crit} . In the case of non-adiabatic mergers, there exists, for certain ranges of the angular momenta, a stable equilibrium position of the supertube. These precisely match the location and constraints on angular momenta of the corresponding static BPS solution. The BMPV angular momentum bound can be violated in a non-adiabatic merger, but only when a barrier is present in the effective potential. One of the significant differences from the two-charge case is that there is a constant magnetic field B_0 along the directions of the compact T^4 , which contributes to the dynamics of the moving supertube. Finally, there is partial, but not exact agreement between the descriptions of adiabatic mergers in the supergravity and worldvolume pictures.

In Section 2 we give background information and definitions. In Section 3 we discuss some basic attributes of a BPS D0-D4-F1 supertube. Section 4 discusses the role of the embedding radius $R(\vec{X})$, as well as adiabatic mergers of the supertube and black hole. In

Section 5 we treat physical scattering, *i.e.* the non-BPS case of slow velocity. In Section 6 we examine conditions for a potential overspin of the black hole in the DBI analysis, and then invoke the supergravity adiabatic merger findings of [11]. We present some brief concluding remarks in Section 7.

2. The BMPV Background

In the ten dimensional type IIA picture the black hole/supertube system has a D0-D4-F1 composition. The full Type IIB supergravity solution for the BMPV metric and other background fields was obtained in [18]; here we describe its IIA counterpart (related by a T-duality transformation on the z direction) using the notation of [2]. It should be noted that the conventions here are related to those of [1] and much of the previous literature by the replacement $\phi_2 \rightarrow -\phi_2$. The D4 branes are wrapped on the compact T^4 , which has volume $V_{T^4} = (2\pi\ell)^4$; the F1 strings are wrapped on the compact z direction, the length of S_z^1 being $2\pi R_z$. The type IIA supergravity solution of the BMPV black hole, with the metric expressed in the string frame, is

$$ds^2 = -H_{D0}^{-1/2}H_{D4}^{-1/2}H_{F1}^{-1}(dt + \gamma_1(\theta)d\phi_1 + \gamma_2(\theta)d\phi_2)^2 + H_{D0}^{1/2}H_{D4}^{1/2}H_{F1}^{-1}dz^2 \quad (2.1)$$

$$+ H_{D0}^{1/2}H_{D4}^{1/2}(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi_1^2 + r^2\cos^2\theta d\phi_2^2) + H_{D0}^{1/2}H_{D4}^{-1/2}ds_{T^4}^2,$$

$$e^{2\Phi} = H_{D0}^{3/2}H_{D4}^{-1/2}H_{F1}^{-1}, \quad (2.2)$$

$$C^{(1)} = (H_{D0}^{-1} - 1)dt + H_{D0}^{-1}(\gamma_1 d\phi_1 + \gamma_2 d\phi_2), \quad (2.3)$$

$$C^{(3)} = -(H_{D4} - 1)r^2\cos^2\theta d\phi_1 \wedge d\phi_2 \wedge dz + H_{F1}^{-1}dt \wedge (\gamma_1 d\phi_1 + \gamma_2 d\phi_2) \wedge dz, \quad (2.4)$$

$$B^{(2)} = (H_{F1}^{-1} - 1)dt \wedge dz + H_{F1}^{-1}(\gamma_1 d\phi_1 + \gamma_2 d\phi_2) \wedge dz, \quad (2.5)$$

where Φ is the dilaton, $B^{(2)}$ is the NS-NS two-form, and $C^{(1)}$ and $C^{(3)}$ are the R-R fields.

The noncompact space we parameterize using $\{x^1, x^2, x^3, x^4\}$ or $\{r, \theta, \phi_1, \phi_2\}$. The angles

satisfy $0 \leq \phi_1, \phi_2 < 2\pi$ and $0 \leq \theta \leq \frac{\pi}{2}$; additionally

$$\tan \phi_1 = \frac{x^2}{x^1}, \quad \tan \phi_2 = \frac{x^4}{x^3}, \quad \text{and} \quad \tan^2 \theta = \frac{(x^1)^2 + (x^2)^2}{(x^3)^2 + (x^4)^2}. \quad (2.6)$$

The angular momentum parameters γ_1, γ_2 and the harmonic functions H_i are given by

$$\gamma_1 = \frac{\omega}{r^2} \sin^2 \theta, \quad \gamma_2 = \frac{\omega}{r^2} \cos^2 \theta, \quad (2.7)$$

$$H_{D0} = 1 + \frac{Q_{D0}}{r^2}, \quad H_{D4} = 1 + \frac{Q_{D4}}{r^2}, \quad H_{F1} = 1 + \frac{Q_{F1}}{r^2}. \quad (2.8)$$

The quantities Q_{D0}, Q_{F1} , and Q_{D4} are the charge parameters of the black hole (defined to be positive), related to the integer numbers of D-branes N_{D0}, N_{F1} , and N_{D4} by

$$N_{D0} = \frac{\ell^4 R_z}{g_s \alpha'^{7/2}} Q_{D0}, \quad N_{F1} = \frac{\ell^4}{g_s^2 \alpha'^3} Q_{F1}, \quad N_{D4} = \frac{R_z}{g_s \alpha'^{3/2}} Q_{D4}, \quad (2.9)$$

where $g_s = e^{\Phi(\vec{r})}|_{r=\infty}$ is the type IIA closed string coupling constant [17]. Since Φ vanishes at infinity in this background, $g_s = 1$; nonetheless we will often keep factors of g_s explicit for clarity. The BMPV black hole is characterized by equal angular momenta in the planes of ϕ_1 and ϕ_2 :

$$J_1 = J_2 = \frac{\pi}{4G_5} \omega \equiv J, \quad (2.10)$$

where G_5 is the gravitational constant in five dimensions. Furthermore, J is bounded:

$$J^2 \leq N_{D0} N_{D4} N_{F1}. \quad (2.11)$$

A violation of this bound would signify the presence of naked closed timelike curves (CTCs).

The field strengths and Bianchi identity are

$$\mathcal{H}^{(3)} = dB^{(2)}, \quad \mathcal{G}^{(2)} = dC^{(1)}, \quad \mathcal{G}^{(4)} = dC^{(3)} + \mathcal{H}^{(3)} \wedge C^{(1)}, \quad (2.12)$$

$$d\mathcal{G}^{(4)} + \mathcal{H}^{(3)} \wedge \mathcal{G}^{(2)} = 0. \quad (2.13)$$

Using $C^{(3)}$ and $C^{(1)}$ it is possible to introduce the “magnetic” potentials $C^{(5)}$ and $C^{(7)}$, and these are necessary in our analysis. They have the following field strengths and Bianchi

identities [19]:¹

$$- * \mathcal{G}^{(4)} = \mathcal{G}^{(6)} = dC^{(5)}, \quad * \mathcal{G}^{(2)} = \mathcal{G}^{(8)} = dC^{(7)} + \mathcal{H}^{(3)} \wedge C^{(5)} \quad (2.14)$$

$$d\mathcal{G}^{(6)} = 0, \quad d\mathcal{G}^{(8)} + \mathcal{H}^{(3)} \wedge \mathcal{G}^{(6)} = 0. \quad (2.15)$$

Notice that the field strength $\mathcal{G}^{(6)}$ is actually the *negative* of $*\mathcal{G}^{(4)}$. The magnetic potentials are found to be

$$C^{(5)} = \left((H_{D4}^{-1} - 1)dt + H_{D4}^{-1}(\gamma_1 d\phi_1 + \gamma_2 d\phi_2) \right) \wedge dT^4 \quad (2.16)$$

$$C^{(7)} = \left(- (H_{D0} - 1)r^2 \cos^2 \theta d\phi_1 \wedge d\phi_2 \wedge dz + H_{F1}^{-1}dt \wedge (\gamma_1 d\phi_1 \wedge dz + \gamma_2 d\phi_2 \wedge dz) \right) \wedge dT^4,$$

where $dT^4 = dx^6 \wedge dx^7 \wedge dx^8 \wedge dx^9$.

3. BPS Three-charge D6 Brane Supertubes

In the D6 brane worldvolume description of [1], the supertube is formed from a D6 brane with four dimensions wrapped on T^4 . Another dimension of the supertube, which we parameterize using σ , wraps a curve S_σ^1 in the uncompactified spacetime and its remaining direction we take to be along the z axis. Thus the worldvolume coordinates are $\{t, \sigma, z, x^6, x^7, x^8, x^9\}$. The D6 brane possess a gauge field on its worldvolume, $F = \frac{1}{2}F_{ab}dx^a \wedge dx^b$. Its general form will be

$$F = F_{tz} dt \wedge dz + F_{\sigma z} d\sigma \wedge dz + F_{t\sigma} dt \wedge d\sigma + F_{67} dx^6 \wedge dx^7 + F_{89} dx^8 \wedge dx^9. \quad (3.1)$$

It should be kept in mind that in our conventions, $F_{t\sigma}$ and $F_{\sigma z}$ have dimensions of length and the other F_{ab} are dimensionless.

The gauge field can be interpreted as a collection of superstrings and lower dimensional D-branes dissolved in the worldvolume of the D6 brane. Thus the supertube carries

¹Reference [19] uses $(+, -, -, \dots, -)$ signature while we use $(-, +, +, \dots, +)$. Therefore the relative signs of the terms in the Bianchi identities and the conventions for the dual fields in [19] are the opposite of ours.

D0 brane charge q_{D0} , D4 brane charge q_{D4} , and fundamental superstring charge q_{F1} , in addition to D2 and D6 brane dipole moments.² The dipole moments are proportional to $n_{D2}R^2$ and $n_{D6}R^2$, where R is the embedding radius discussed in Section 4, and n_{D2} and n_{D6} are the so-called “dipole charges” (see *e.g.* [6]).

Our conventions for embedding the worldvolume coordinates into the spacetime (*i.e.* our gauge choice for the Lagrangian) are that we align the axes of $\{t, z, x^6, x^7, x^8, x^9\}$ with those of the spacetime (the static gauge). The position of the supertube in the noncompact space is given by the coordinates $X^i = \{r, \theta, \phi_1, \phi_2\}$. Adopting the translational invariance of [1, 2], we let X^i depend on σ and t , but not on z or the T^4 directions, while F_{ab} depends only on t . $F_{\sigma z}$ is positive in the BPS configuration, and we will take $F_{\sigma z} > 0$ throughout the paper. Moreover, we consider only the simplest circular embedding $\phi_1 = \sigma$. The embedding is treated in further detail in Section 4.

In the worldvolume description supertubes can be discussed in terms of the DBI action (using units in which $2\pi\alpha' = 1$),

$$\begin{aligned} S &= \int L dt = \int \mathcal{L} d^7x = S_{DBI} + S_{WZ} \\ &= -\tau_{D6} \int d^7x e^{-\Phi} \sqrt{-\det(g_{ab} + b_{ab} + F_{ab})} + \tau_{D6} g_s \int \sum_{7\text{-forms}} c^{(m)} \wedge e^{(F+b)^{(2)}}. \end{aligned} \quad (3.2)$$

The lower case variables (g_{ab} , b_{ab} , and $c^{(m)}$) refer to the pullbacks of the spacetime fields $G_{\mu\nu}$, $B_{\mu\nu}$, and $C^{(m)}$ to the D6 worldvolume. The Wess-Zumino (WZ) integral is over the D6 brane worldvolume and so the sum over m only includes terms in the wedge product that are 7-forms. Setting $g_s = 1$, the WZ term thus expands as

$$\begin{aligned} S_{WZ} &= \tau_{D6} \int \left(c^{(7)} + c^{(5)} \wedge (F+b)^{(2)} + \frac{1}{2!} c^{(3)} \wedge (F+b)^{(2)} \wedge (F+b)^{(2)} \right. \\ &\quad \left. + \frac{1}{3!} c^{(1)} \wedge (F+b)^{(2)} \wedge (F+b)^{(2)} \wedge (F+b)^{(2)} \right). \end{aligned} \quad (3.3)$$

²The most general three-charge supertube also has an NS5 brane dipole moment, but that dipole is not captured by a worldvolume treatment.

The explicit Lagrangian appears in Section 5 and Appendix A. The BPS limit is specified by

$$F_{tz} = 1, \quad F_{t\sigma} = 0, \quad F_{67} = F_{89}, \quad \text{and} \quad \partial_t X^i = 0. \quad (3.4)$$

This provides a symmetry under exchange of the pair $\{X^6, X^7\}$ with $\{X^8, X^9\}$. Throughout the paper we will restrict ourselves to the case $F_{67} = F_{89}$. For the case of $F_{67} = F_{89} = B_0$, the supertube charges take the form

$$q_{D4} = \frac{\tau_{D6}}{\tau_{D4}} \int dz d\sigma F_{\sigma z} = \frac{R_z}{\alpha'} F_{\sigma z}, \quad (3.5)$$

$$\begin{aligned} q_{D2} &= \frac{\tau_{D6}}{\tau_{D2}} \int dz d\sigma (dX^6 dX^7 F_{\sigma z} F_{67} + dX^8 dX^9 F_{\sigma z} F_{89}) \\ &= 2 \frac{\ell^2 R_z}{\alpha'^2} F_{\sigma z} B_0, \end{aligned} \quad (3.6)$$

$$q_{D0} = \frac{\tau_{D6}}{\tau_{D0}} \int dT^4 dz d\sigma F_{\sigma z} F_{67} F_{89} = \frac{\ell^4 R_z}{\alpha'^3} F_{\sigma z} B_0^2, \quad (3.7)$$

$$q_{F1} = \frac{1}{\tau_{F1}} \int dT^4 d\sigma \frac{\partial \mathcal{L}}{\partial F_{tz}}. \quad (3.8)$$

These normalizations ensure that the charges take integer values. Since

$$\frac{q_{D0}}{q_{D4}} = \frac{\ell^4}{\alpha'^2} B_0^2, \quad (3.9)$$

all expressions involving B_0^2 can also be written in terms of the ratio of supertube charges.

It is noteworthy that taking $B_0 \rightarrow 0$ leaves us with another type of two-charge supertube: a D6 brane supertube with D4 and F1 charge. This is T-dual to the D0-F1 supertube of [3, 4], and its physical behavior is easily obtained from that of the D0-F1 supertube using the relations of Appendix A. We add parenthetically that there are also two-charge supertubes characterized by nonzero q_{D0}, q_{D4} and n_{NS5} [1].

The tensions $\tau_{D0}, \tau_{D2}, \tau_{D4}$ and τ_{D6} of the Dp-branes take the form

$$\tau_{Dp} = \frac{1}{(2\pi)^p g_s \alpha'^{(p+1)/2}}, \quad (3.10)$$

while the the tension τ_{F1} of the F-strings is $\frac{1}{2\pi\alpha'}$. Unlike the other charges, q_{F1} has a value that in general depends on the position and velocity of the supertube, as we will see in

Section 5. Its form in a BPS configuration is

$$q_{F1} = 2\pi V_{T^4} \frac{\tau_{D6} (H_{D0} + B_0^2 H_{D4}) r^2 \sin^2 \theta}{\tau_{F1} F_{\sigma z}} = \frac{\ell^4}{g_s \alpha'^{5/2}} \frac{(H_{D0} + B_0^2 H_{D4}) r^2 \sin^2 \theta}{F_{\sigma z}}. \quad (3.11)$$

It is significant that with the black hole present the functional form of q_{F1} depends on position; this is the reason for a limited range of allowed locations for a given BPS supertube, and does not occur in flat spacetime.

Our interest is in a supertube that will be T-dual to a D1-D5-P configuration, so we require it to carry D0, D4, and F1 charge, but no D2 charge. Similarly we require it to possess D2 and D6 dipole charge but no D4 dipole charge. Eq. (3.7) shows that a single D6 brane tube is inappropriate for such a task. Consequently, we construct the supertube out of an even number k of coincident D6 branes that are expected [1, 7] to form a marginally bound state³. (This is done with the understanding that k is small enough for the DBI approximation to hold.) Half of these D6 branes have $F_{67} = F_{89} = -B_0$, and thus have the opposite sign of q_{D2} , but are otherwise identical to the rest.⁴ The F_{ab} become diagonal $k \times k$ matrices \mathcal{F}_{ab} (which, naturally, commute) and total charge is given by tracing over the matrices. Such a configuration has $\mathcal{F}_{67} = \mathcal{F}_{89}$; $\text{Tr } \mathcal{F}_{67} = \text{Tr } \mathcal{F}_{89} = 0$; and \mathcal{F}_{tz} , $\mathcal{F}_{\sigma z}$, $\mathcal{F}_{67}\mathcal{F}_{89}$ all proportional to the unit matrix.

The net D2 charge is eliminated because $\mathcal{F}_{\sigma z}\mathcal{F}_{67}$ and $\mathcal{F}_{\sigma z}\mathcal{F}_{89}$ have vanishing trace, and analysis of the open string spectrum indicates that there is no danger of a tachyon instability from the dissolved D2 and $\overline{D2}$ branes [14]. Meanwhile the D0, D4, and F1

³Actually demonstrating that a such a state is in fact bound, *i.e.* has a discrete energy spectrum, is nontrivial. It has been achieved for the two-charge supertube in [13].

⁴The method of [1] does not require the branes to lie in the same plane or have the same size in flat space. However, we want to ensure that the bound state is maintained in the vicinity of the black hole. Since there is no binding energy between the D6 branes, the scattering behavior of all k branes must be identical. This can be achieved if the branes all have the same embedding and physically differ by no more than the sign of B_0 , since, as we will see, the dynamics depends on B_0^2 rather than B_0 itself.

charges merely obtain a factor of k :

$$\mathfrak{q}_{D4} = \frac{\tau_{D6}}{\tau_{D4}} V_2 \operatorname{Tr} \mathcal{F}_{\sigma z} = k \frac{R_z}{\alpha'} F_{\sigma z}, \quad (3.12)$$

$$\begin{aligned} \mathfrak{q}_{D2} &= \frac{R_z \ell^2}{\alpha'^2} (\operatorname{Tr}(\mathcal{F}_{\sigma z} \mathcal{F}_{67}) + \operatorname{Tr}(\mathcal{F}_{\sigma z} \mathcal{F}_{89})) \\ &= \frac{R_z \ell^2}{\alpha'^2} F_{\sigma z} (\operatorname{Tr} \mathcal{F}_{67} + \operatorname{Tr} \mathcal{F}_{89}) = 0, \end{aligned} \quad (3.13)$$

$$\mathfrak{q}_{D0} = \frac{\tau_{D6}}{\tau_{D0}} V_6 \operatorname{Tr}(\mathcal{F}_{\sigma z} \mathcal{F}_{67} \mathcal{F}_{89}) = k \frac{\ell^4 R_z}{\alpha'^3} F_{\sigma z} B_0^2, \quad (3.14)$$

$$\mathfrak{q}_{F1} = \frac{1}{\tau_{F1}} \int dT^4 d\sigma \operatorname{Tr} \left(\frac{\partial \mathcal{L}}{\partial F_{tz}} \right). \quad (3.15)$$

Here V_2 is the two-volume $(2\pi)^2 R_z$ over $\{\sigma, z\}$, V_6 is the full spatial six-volume $(2\pi)^6 R_z \ell^4$, and we note that each charge takes positive integer values. For variables such as charge, angular momentum, and dipole charge, we use Fraktur letters ($\mathfrak{q}, \mathfrak{j}, \mathfrak{n}$) to denote quantities that describe the supertube as a whole, and italic type (q, j, n) for those that correspond to one of the constituent branes. At times we will label the charges as $\{\mathfrak{q}_I\}$ where $I = 1, 2, 3$ and $\{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3\} = \{\mathfrak{q}_{D0}, \mathfrak{q}_{F1}, \mathfrak{q}_{D4}\}$.

The dipole charges \mathfrak{n} , expressed in units in which they take integer values, take the form

$$\mathfrak{n}_{D6} = \operatorname{Tr}(I_k) = k, \quad (3.16)$$

$$\begin{aligned} \mathfrak{n}_{D4} &= \frac{\tau_{D6}}{\tau_{D4}} \frac{V_6}{(2\pi\ell)^2 V_2} (\operatorname{Tr} \mathcal{F}_{67} + \operatorname{Tr} \mathcal{F}_{89}) \\ &= \frac{\ell^2}{\alpha'} (\operatorname{Tr} \mathcal{F}_{67} + \operatorname{Tr} \mathcal{F}_{89}) = 0, \end{aligned} \quad (3.17)$$

$$\mathfrak{n}_{D2} = \frac{\tau_{D6}}{\tau_{D2}} \frac{V_6}{V_2} \operatorname{Tr}(\mathcal{F}_{67} \mathcal{F}_{89}) = k \frac{\ell^4}{\alpha'^2} B_0^2, \quad (3.18)$$

$$\mathfrak{n}_{NS5} = 0, \quad (3.19)$$

where I_k is the unit $k \times k$ matrix. The vanishing of \mathfrak{n}_{D4} is closely related to the vanishing of \mathfrak{q}_{D2} as both are proportional to $(\operatorname{Tr} \mathcal{F}_{67} + \operatorname{Tr} \mathcal{F}_{89})$. The final dipole charge \mathfrak{n}_{NS5} is not captured in a worldvolume treatment [1], and is set to zero, leaving us with three nonzero charges and two dipole charges. The case of a three-charge supertube with only one dipole

charge, as well as that of two charges and two dipoles, is pathological and always contains CTCs [15].

In our case, for the geometry near the supertube to be free of CTCs we need [15, 6]

$$\frac{\mathbf{n}_{D2}}{\mathbf{n}_{D6}} = \frac{\mathbf{q}_{D0}}{\mathbf{q}_{D4}} \quad \rightarrow \quad \frac{\mathbf{n}_{D2}}{\mathbf{n}_{D6}} = \frac{q_{D0}}{q_{D4}} = \frac{\ell^4}{\alpha'^2} B_0^2, \quad (3.20)$$

which does indeed follow from the preceding equations. There are interesting implications. If k and q_{D4} have no common divisors, (3.20) then implies that $\frac{\ell^4}{\alpha'^2} B_0^2$ is also an integer, and that \mathbf{n}_{D2} and \mathbf{q}_{D0} are integer multiples of \mathbf{n}_{D6} and \mathbf{q}_{D4} , respectively. This dovetails with the statement in [1] that this supertube can be viewed as a superposition of \mathbf{n}_{D2} ordinary D0-F1 supertubes and \mathbf{n}_{D6} D4-F1 supertubes if the tubes all have the same radius, all coincide and (3.20) is satisfied. Even in the case that k and q_{D4} do have common divisors, B_0 is constrained because $\frac{\ell^4}{\alpha'^2} B_0^2$ must be a rational number. Furthermore, the values of q_{D0} and \mathbf{n}_{D2} are somewhat restricted because arbitrary combinations do not satisfy (3.20) and thus do not lead to regular geometries.

Under this construction, the pullbacks g_{ab} , b_{ab} , and $c_{ab}^{(m)}$ also become multiples of the $k \times k$ identity matrix, denoted by \mathfrak{g}_{ab} , \mathfrak{b}_{ab} , and $\mathfrak{c}_{ab}^{(m)}$. The action is now

$$S = -\tau_{D6} \int d^7x e^{-\Phi} \text{Tr} \left[\sqrt{-\det(\mathfrak{g}_{ab} + \mathfrak{b}_{ab} + \mathcal{F}_{ab})} \right] + \tau_{D6} \int \text{Tr} \left[\sum_m \mathfrak{c}^{(m)} \wedge e^{(\mathcal{F} + \mathfrak{b})^{(2)}} \right]. \quad (3.21)$$

After tracing over the matrices in the action, the energy, angular momenta and other physical quantities of the supertube are obtained in the usual way.

The angular momenta of the supertube in the ϕ_1 and ϕ_2 directions are

$$j_1 = k j_1 = \left| \int dz d\sigma dT^4 \frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} \right| = k q_{F1} q_{D4}, \quad (3.22)$$

$$j_2 = k j_2 = \left| \int dz d\sigma dT^4 \frac{\partial \mathcal{L}}{\partial \dot{\phi}_2} \right|. \quad (3.23)$$

For general D6 brane supertubes (two-charge and three-charge), there is a bound $0 < j_1 \leq q_{F1} q_{D4}$. Here we saturate the bound because our embedding is circular and F_{ab} is

homogeneous on the worldvolume of the D6 brane(s) [5]. Meanwhile the supersymmetric value of j_2 is given by

$$j_2 = j_{crit} \cos^2 \theta, \quad \text{where} \quad (3.24)$$

$$j_{crit} = \tau_{D6} V_6 (Q_{D0} + B_0^2 Q_{D4}) = N_{D0} + B_0^2 \frac{\ell^4}{\alpha'^2} N_{D4} = N_{D0} + \frac{q_{D0}}{q_{D4}} N_{D4}.$$

Note that $j_1 = \mathfrak{q}_{F1} \mathfrak{q}_{D4} / k$. Significantly, j_1 depends on neither the black hole charges nor the dimensions of the compact T^4 while j_2 depends on both. In fact, j_2 vanishes in the absence of the black hole⁵, while j_1 represents angular momentum along the circumference of the supertube. The circumferential angular momentum j_1 is essential for the stability of the supertube against collapse [3], and so $j_1 > 0$ always holds.

The supersymmetric value of the action is

$$S = -\tau_{D6} k \int dt d\sigma dz dT^4 F_{\sigma z} (1 + B_0^2) = - \int dt (\tau_{D0} \mathfrak{q}_{D0} + V_{T^4} \tau_{D4} \mathfrak{q}_{D4}). \quad (3.25)$$

This leads us to the energy of the supertube, constructed from the Hamiltonian $\sum \tilde{p}_i \dot{q}_i - L$, giving

$$E = \frac{\partial L}{\partial F_{tz}} F_{tz} - L = 2\pi R_z \tau_{F1} \mathfrak{q}_{F1} + \tau_{D0} \mathfrak{q}_{D0} + V_{T^4} \tau_{D4} \mathfrak{q}_{D4}. \quad (3.26)$$

This is a minimum which saturates a BPS bound.

4. The Embedding Radius and Adiabatic Mergers

4.1 The Embedding Radius

It is illuminating to examine the embedding radius of the supertube in the spacetime. Our choice of embedding was to let the directions $\{t, z, x^6, x^7, x^8, x^9\}$ of the worldvolume coincide with those of the spacetime, take the embedding coordinates

$$\{X^i\} = \{t, r, \theta, \phi_1, \phi_2, Z, X^6, X^7, X^8, X^9\} \quad (4.1)$$

⁵This stands in contrast to the case of supergravity supertubes, in which j_2 is present regardless.

to be independent of z and the T^4 directions, and set

$$\phi_1 = \sigma. \quad (4.2)$$

This choice exhausts our reparameterization freedom and thus represents a physical choice of orientation of the supertube with respect to the black hole. The points of the supertube all have the same values of the coordinates $\{r, \theta, \phi_2\}$ and differ in their values of ϕ_1 and $\{Z, X^6, X^7, X^8, X^9\}$. The latter set we take to be independent of t and σ , so the relevant embedding coordinates are $\{r = r(t), \theta = \theta(t), \phi_1 = \sigma, \phi_2 = \phi_2(t)\}$, and there is no motion in the ϕ_1 direction. We will also make some use of the coordinates $\rho_1 = r \sin \theta, \rho_2 = r \cos \theta$; the cross section of the supertube lies in the ρ_1 - θ_1 plane.

The worldvolume interval is given by

$$ds^2 = g_{ab} dx^a dx^b = g_{tt} dt^2 + 2g_{t\sigma} dt d\sigma + g_{\sigma\sigma} d\sigma^2 + g_{zz} dz^2 + ds_{T^4}^2 \quad (4.3)$$

Now, g_{ab} is the pullback of the spacetime metric $G_{\mu\nu}$ to the worldvolume of the supertube; with the embedding (4.2) its components are

$$\begin{aligned} g_{tt} &= -H_{D0}^{-1/2} H_{D4}^{-1/2} H_{F1}^{-1} \left(1 + \frac{\omega}{r^2} \cos^2 \theta \dot{\phi}_2\right)^2 + H_{D0}^{1/2} H_{D4}^{1/2} (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \cos^2 \theta \dot{\phi}_2^2), \\ g_{t\sigma} &= -H_{D0}^{-1/2} H_{D4}^{-1/2} H_{F1}^{-1} \left(\frac{\omega}{r^2} \sin^2 \theta + \frac{\omega^2}{r^4} \sin^2 \theta \cos^2 \theta \dot{\phi}_2\right), \\ g_{\sigma\sigma} &= H_{D0}^{1/2} H_{D4}^{1/2} r^2 \sin^2 \theta - H_{D0}^{-1/2} H_{D4}^{-1/2} H_{F1}^{-1} \frac{\omega^2}{r^4} \sin^4 \theta, \\ g_{zz} &= H_{D0}^{1/2} H_{D4}^{1/2} H_{F1}^{-1}, \\ g_{tz} &= g_{\sigma z} = 0, \\ g_{66} &= g_{77} = g_{88} = g_{99} = H_{D0}^{1/2} H_{D4}^{-1/2}, \end{aligned} \quad (4.4)$$

keeping in mind that the metric is in the string frame and ω is the angular momentum parameter of the black hole from (2.10). In the BPS limit, the time derivatives vanish.

Furthermore, we can change bases by switching from $\{dt, d\sigma\}$ to $\{e^0, d\tilde{\sigma}\}$ where $e^0 = dt + \frac{\omega}{r^2} \sin^2 \theta d\sigma$ and $d\tilde{\sigma} = d\sigma$, obtaining

$$\begin{aligned} ds^2 &= -H_{D0}^{-1/2} H_{D4}^{-1/2} H_{F1}^{-1} (e^0)^2 + H_{D0}^{1/2} H_{D4}^{1/2} r^2 \sin^2 \theta d\tilde{\sigma}^2 + H_{D0}^{1/2} H_{D4}^{1/2} H_{F1}^{-1} dz^2 + ds_{T^4}^2 \\ &\equiv -H_{D0}^{-1/2} H_{D4}^{-1/2} H_{F1}^{-1} (e^0)^2 + R^2 d\tilde{\sigma}^2 + H_{D0}^{1/2} H_{D4}^{1/2} H_{F1}^{-1} dz^2 + ds_{T^4}^2. \end{aligned} \quad (4.5)$$

It is perhaps useful to point out that $dt \rightarrow dt + \frac{\omega}{r^2} \sin^2 \theta$ is not a globally well defined coordinate transformation, due to the fact that the S_σ^1 is noncontractible. This prevents us from introducing a coordinate \tilde{t} such that⁶ $d\tilde{t} = e^0$.

The embedding radius $R = R(\vec{X})$ of the supertube is just given by $R^2 = g_{\tilde{\sigma}\tilde{\sigma}}$. Thus,

$$R^2 = H_{D0}^{1/2} H_{D4}^{1/2} r^2 \sin^2 \theta = (Q_{D0} + r^2)^{1/2} (Q_{D4} + r^2)^{1/2} \sin^2 \theta. \quad (4.6)$$

Combining (3.5), (3.11) and (3.24) gives⁷

$$j_1 = q_{F1} q_{D4} = \tau_{D6} V_6 (Q_{D0} + r^2 + B_0^2 (Q_{D4} + r^2)) \sin^2 \theta. \quad (4.7)$$

In the flat spacetime limit $Q_I \rightarrow 0$, this reproduces Eq. (8.2) of [6] after appropriate redefinitions.

Since we are required to have $j_1 > 0$, (4.7) tells us that there are no BPS solutions of the DBI action for $\theta = 0$ at finite r , and that the allowed supersymmetric values of r are given by

$$r^2 = \frac{j_1 - j_{crit} \sin^2 \theta}{\tau_{D6} V_6 (1 + B_0^2) \sin^2 \theta}. \quad (4.8)$$

Since we know from (3.24) that $\cos^2 \theta = j_2 / j_{crit}$ we see that

$$r^2 = \frac{1}{\tau_{D6} V_6 (1 + B_0^2)} \frac{(j_1 + j_2 - j_{crit}) j_{crit}}{j_{crit} - j_2}. \quad (4.9)$$

⁶We thank D. Marolf for correspondence on this point.

⁷In the next two sections, many of the expressions are given in terms of (q, j) rather than (\mathbf{q}, \mathbf{j}) to make manifest the fact that they are independent of k , the number of constituent branes.

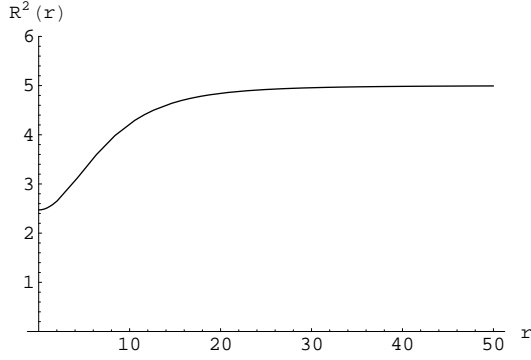


Figure 1: The embedding radius as a function of r ($\tau_{D6} = 1, V_6 = \frac{1}{2}, B_0 = 1, j_1 = 5, Q_{D0} = 21, Q_{D4} = 300$).

Supertube locations must satisfy $r > 0$, as those precisely at $r = 0$ would have null worldvolume [2]. Eq. (4.9) then tells us that BPS supertubes must satisfy

$$j_{crit} - j_1 < j_2 < j_{crit}. \quad (4.10)$$

For the embedding radius we can use (4.8) and (4.9) to obtain expressions that depend solely on θ , or solely on r , that are conducive to taking limits:

$$R^2 = \frac{\left(j_1 + (N_{D0} - \frac{\ell^4}{\alpha^2} N_{D4}) B_0^2 \sin^2 \theta\right)^{1/2} \left(j_1 - (N_{D0} - \frac{\ell^4}{\alpha^2} N_{D4}) \sin^2 \theta\right)^{1/2}}{\tau_{D6} V_6 (1 + B_0^2)}, \quad (4.11)$$

$$= (Q_{D0} + r^2)^{1/2} (Q_{D4} + r^2)^{1/2} \frac{j_1}{\tau_{D6} V_6 (Q_{D0} + B_0^2 Q_{D4} + (1 + B_0^2) r^2)}. \quad (4.12)$$

As Figure 1 shows, the latter is a monotonically increasing function of r . The corresponding relations for the D0-F1 tube are obtained using the formulae of Appendix B. We see that $r \rightarrow \infty$ is the same limit as $\sin^2 \theta \rightarrow 0$, and in this limit

$$R^2 \rightarrow \frac{j_1}{\tau_{D6} V_6 (1 + B_0^2)} \equiv R_\infty^2. \quad (4.13)$$

This is independent of the black hole charges, as we might expect. In fact, it is precisely the value obtained for a supertube in a Minkowski background.

The BPS configuration space is best visualized using the coordinates

$$\rho_1 = r \sin \theta = \sqrt{(X^1)^2 + (X^2)^2}, \quad \rho_2 = r \cos \theta = \sqrt{(X^3)^2 + (X^4)^2}. \quad (4.14)$$

From (4.8) we see that ρ_1 has a maximum value of

$$\rho_1^{max} = \sqrt{\frac{j_1}{\tau_{D6} V_6 (1 + B_0^2)}}, \quad (4.15)$$

and that large distances from the black hole imply large values of ρ_2 , as shown in Figures 2 and 3.

4.2 Adiabatic Mergers

The BPS configurations are static, and for a given j_1 and j_2 there is only one (r, θ) location of the supertube (ϕ_2 , on the other hand, is completely unrestricted). Thus the authors of [1], [11], and [12] consider a scenario in which j_2 is allowed to vary, so that the supertube can explore the BPS configuration space, moving at infinitesimal velocity. This of course requires the application of an external torque on the supertube, but the energy changes only infinitesimally and $\{j_1, \mathbf{q}_{D0}, \mathbf{q}_{F1}, \mathbf{q}_{D4}, \mathbf{n}_{D2}, \mathbf{n}_{D6}\}$ are conserved⁸. They then invoke the adiabatic limit of vanishing velocity.

Of particular interest are the BPS solutions in which the ring is arbitrarily close to the black hole horizon. The existence of these solutions gives rise to the idealized process of using the adiabatic limit to bring the supertube to the horizon and then infinitesimally farther, allowing it to fall into the black hole: an “adiabatic merger”. Throughout the merger, the system is treated as though it were a BPS configuration. The fact that there are no BPS solutions at the horizon itself further distances this merger from an actual physical process, however. For actual physical motion, analysis of the full non-BPS Lagrangian is needed. This will be treated in Section 5.

Since $\sin^2 \theta \leq 1$ we can see from (4.8) that it is possible to adiabatically bring the supertube to the horizon $r = 0$ if

$$j_1 \leq j_{crit} = N_{D0} + B_0^2 \frac{\ell^4}{\alpha'^2} N_{D4}, \quad (4.16)$$

⁸Conservation of these quantities is discussed in more detail in Section 5.

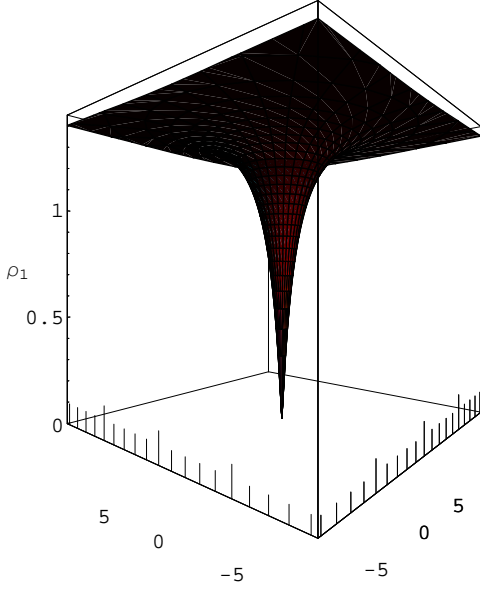


Figure 2: The configuration space (in cylindrical coordinates $\{\rho_2, \phi_2, \rho_1\}$ where ρ_1 is on the vertical axis) of BPS solutions for the location of a supertube for a given value of j_1 when $j_1 \leq j_{crit}$ ($\tau_{D6} = 1, V_6 = \frac{1}{2}, B_0 = 1, j_1 = 2, j_{crit} = 3; j_2$ is not fixed). Solutions exist for all $\{\rho_1, \rho_2\}$ such that $0 < \rho_1 < \sqrt{2}$ and $\rho_2 > 0$.

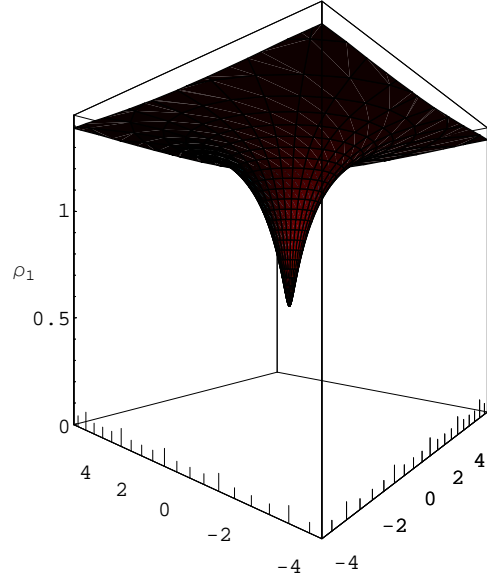


Figure 3: The configuration space of BPS solutions when $j_1 > j_{crit}$. There are no BPS solutions for $\rho_1 < r_{min}$ ($\tau_{D6} = 1, V_6 = \frac{1}{2}, B_0 = 1, j_1 = 2, j_{crit} = 1.7, r_{min} = 0.55$). Again, j_2 is not fixed.

as in Figure 2. When it reaches the horizon, its size will have decreased to

$$R^2 = \frac{j_1}{j_{crit}} Q_{D0}^{1/2} Q_{D4}^{1/2} \equiv R_0^2, \quad (4.17)$$

and the merging value of θ is just given by $\sin^2 \theta_{merge} = j_1/j_{crit}$. On the other hand, if $j_1 > j_{crit}$, then the supertube cannot reach $r = 0$ (because it cannot reach $\rho_1 = 0$); the closest it can come is

$$r^2 = \frac{j_1 - j_{crit}}{\tau_{D6} V_6 (1 + B_0^2)} \equiv r_{min}^2. \quad (4.18)$$

This minimum distance occurs when $\sin \theta = 1$, so $\rho_1 = r = r_{min}$ and $\rho_2 = 0$, as shown in Figure 3. The embedding radius at r_{min} is

$$R^2 = (Q_{D0} + r_{min}^2)^{1/2} (Q_{D4} + r_{min}^2)^{1/2} > R_0^2. \quad (4.19)$$

Now, near the horizon, the BMPV metric looks like $AdS_2 \times S^3$. In [1] it was pointed out that the S^3 has a radius (in the string frame) given by

$$R_{S^3}^2 = Q_{D0}^{1/2} Q_{D4}^{1/2}. \quad (4.20)$$

Thus if the embedding radius of a BPS supertube satisfies $R^2 > Q_{D0}^{1/2} Q_{D4}^{1/2}$, it was argued that the supertube will not “fit” inside the near horizon region of the black hole; this explains why the supertube cannot adiabatically merge with it. This reasoning holds here as well, for the case of adiabatic mergers. However, we are not aware of an extension of this argument to the case of a non-adiabatic merger, in which the tube has no obvious counterpart to R . Indeed, it turns out that a supertube that is naively “too big” to fit *can* be pushed into the black hole (taking us into the non-adiabatic regime), as we will discuss shortly.

For the two charge cases it happens that $j_{crit} = N_{D4}$ for the D0-F1 supertube, and $j_{crit} = N_{D0}$ for the D4-F1 supertube. The three-charge supertube differs crucially in that j_{crit} depends on $\frac{\ell^4}{\alpha'^2} B_0^2 = q_{D0}/q_{D4}$. We are led to the intriguing result that the volume of the compact T^4 and its magnetic field affect whether or not a given three-charge supertube can merge with the black hole.

5. The Scattering Calculation: Low Velocity Non-adiabatic Mergers

Moving away from the idealized limit of an adiabatic merger, we consider the case of the supertube moving at finite but slow velocity in the BMPV background. Here we are not assuming the presence of any external torque, so, unlike the adiabatic case, j_2 is conserved. The motion produces small deviations from the BPS configuration, thus breaking all supersymmetries, and the energy of the tube increases to $E_{BPS} + \Delta E$. To calculate ΔE , we will expand \mathcal{L} to second order in the velocities $\partial_t X^i$ and the fields $F_{t\sigma}$

and $\delta F_{tz} \equiv F_{tz} - 1$. With our embedding the expanded Lagrangian density is

$$\begin{aligned}
(\tau_{D6}k)^{-1}\mathcal{L} = & -F_{\sigma z}(1+B_0^2) + (Q_{D0} + B_0^2Q_{D4})\cos^2\theta\dot{\phi}_2 + \frac{(H_{D0} + B_0^2H_{D4})}{2F_{\sigma z}}F_{t\sigma}^2 \\
& + \frac{(H_{D0} + B_0^2H_{D4})r^2\sin^2\theta}{F_{\sigma z}}\delta F_{tz} + \left(\frac{F_{\sigma z}H_{F1}}{2} + \frac{\omega\sin^2\theta}{r^2} + \frac{H_{D0}H_{D4}r^2\sin^2\theta}{2F_{\sigma z}}\right) \\
& \times (H_{D0} + B_0^2H_{D4})\left(\frac{r^2\sin^2\theta}{F_{\sigma z}^2}\delta F_{tz}^2 + \dot{r}^2 + r^2\dot{\theta}^2 + r^2\cos^2\theta\dot{\phi}_2^2\right). \tag{5.1}
\end{aligned}$$

The charge $\mathbf{q}_{F1} = k q_{F1}$ becomes

$$\mathbf{q}_{F1} = 2\pi k V_{T^4} \frac{\tau_{D6}}{\tau_{F1}} \frac{(H_{D0} + B_0^2H_{D4})r^2\sin^2\theta}{F_{\sigma z}} \left(1 + 2\frac{\delta F_{tz}}{F_{\sigma z}} \left(\frac{F_{\sigma z}H_{F1}}{2} + \frac{\omega\sin^2\theta}{r^2} + \frac{H_{D0}H_{D4}r^2\sin^2\theta}{2F_{\sigma z}}\right)\right) \tag{5.2}$$

and the angular momenta are computed to be

$$\mathbf{j}_1 = k j_1 = k q_{F1} q_{D4}, \quad \text{and} \tag{5.3}$$

$$\mathbf{j}_2 = k j_2 = \tau_{D6} V_6 k \cos^2\theta \tag{5.4}$$

$$\times \left[(Q_{D0} + B_0^2Q_{D4}) + \frac{(H_{D0} + B_0^2H_{D4})\dot{\phi}_2}{F_{\sigma z}} (F_{\sigma z}^2 H_{F1} r^2 + 2\omega F_{\sigma z} \sin^2\theta + H_{D0}H_{D4}r^4 \sin^2\theta) \right].$$

Recalling that F_{ab} is independent of all worldvolume variables but t , the equations of motion for F_{ab} become

$$0 = \partial_a \frac{\partial \mathcal{L}}{\partial F_{ab}} = \partial_t \frac{\partial \mathcal{L}}{\partial F_{tb}}. \tag{5.5}$$

where $a, b = \{t, \sigma, z, x^6, \dots, x^9\}$. For $b = z$ this leads to the conservation of \mathbf{q}_{F1} . When $b = \sigma$ (5.5) implies that

$$(H_{D0} + B_0^2H_{D4})F_{t\sigma} = \text{const}. \tag{5.6}$$

Thus there are trajectories in which the supersymmetric value of $F_{t\sigma}$, namely $F_{t\sigma} = 0$, is maintained throughout the motion; these are the ones considered in the analysis below, although for now we keep $F_{t\sigma}$ general. The field F_{ab} also satisfies the Bianchi identity

$dF = 0$, whence the relations

$$\partial_t F_{\sigma z} + \partial_z F_{t\sigma} + \partial_\sigma F_{zt} = 0 \rightarrow \partial_t F_{\sigma z} = 0, \quad (5.7)$$

$$\partial_t F_{67} + \partial_{x^7} F_{t6} + \partial_{x^6} F_{7t} = 0 \rightarrow \partial_t F_{67} = 0, \quad (5.8)$$

$$\partial_t F_{89} + \partial_{x^9} F_{t8} + \partial_{x^8} F_{9t} = 0 \rightarrow \partial_t F_{89} = 0. \quad (5.9)$$

Eq. (5.7) leads to the conservation of \mathfrak{q}_{D4} ; (5.8) and (5.9) lead to conservation of \mathfrak{q}_{D0} and \mathfrak{q}_{D2} . It follows from (3.16) and (3.20) that the dipole charges \mathfrak{n}_{D2} and \mathfrak{n}_{D6} remain constant, although for a generic supergravity supertube they are not conserved. Naturally, conservation of j_1 and j_2 follow from the equations of motion for ϕ_1 and ϕ_2

$$\partial_t \frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} = 0, \quad \partial_t \frac{\partial \mathcal{L}}{\partial \dot{\phi}_2} = 0, \quad (5.10)$$

where we have momentarily allowed ϕ_1 to depend on time.

Once again setting $F_{t\sigma} = \dot{\phi}_1 = 0$, the energy takes the form

$$\begin{aligned} E &= F_{tz} \frac{\partial L}{\partial F_{tz}} + \dot{r} \frac{\partial L}{\partial \dot{r}} + \dot{\theta} \frac{\partial L}{\partial \dot{\theta}} + \dot{\phi}_2 \frac{\partial L}{\partial \dot{\phi}_2} - L \\ &= 2\pi R_z \tau_{F1} \mathfrak{q}_{F1} + \tau_{D0} \mathfrak{q}_{D0} + V_{T^4} \tau_{D4} \mathfrak{q}_{D4} + \Delta E. \end{aligned} \quad (5.11)$$

After using (5.2) and (5.4) to eliminate δF_{tz} and $\dot{\phi}_2$ for the conserved quantities q_{F1} and j_2 , ΔE is given by

$$\begin{aligned} \Delta E &= \quad (5.12) \\ \tau_{D6} V_6 k &\left[\frac{F_{\sigma z}}{2(H_{D0} + B_0^2 H_{D4})(F_{\sigma z}^2 H_{F1} r^2 + H_{D0} H_{D4} r^4 \sin^2 \theta + 2\omega F_{\sigma z} \sin^2 \theta)} \right. \\ &\quad \times \left(\frac{[j_1/(\tau_{D6} V_6) - (H_{D0} + B_0^2 H_{D4}) r^2 \sin^2 \theta]^2}{\sin^2 \theta} + \frac{[j_2/(\tau_{D6} V_6) - (Q_{D0} + B_0^2 Q_{D4}) \cos^2 \theta]^2}{\cos^2 \theta} \right) \\ &\quad \left. + \frac{(H_{D0} + B_0^2 H_{D4})(F_{\sigma z}^2 H_{F1} r^2 + H_{D0} H_{D4} r^4 \sin^2 \theta + 2\omega F_{\sigma z} \sin^2 \theta)}{2F_{\sigma z} r^2} (\dot{r}^2 + r^2 \dot{\theta}^2) \right]. \end{aligned}$$

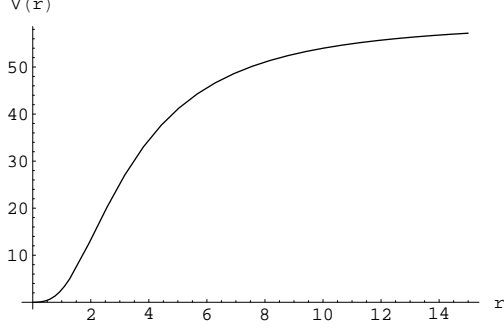


Figure 4: When $j_1 < j_{crit}$ there is no local potential minimum. Above, $j_1 = 1$ and $j_{crit} = 2$. In both figures, $\tau_{D6} = V_6 = Q_{D4} = Q_{F1} = Q_{D0} = B_0 = 1$, $k = 20$, $F_{\sigma z} = 3$, and $\omega = \frac{1}{2}$.

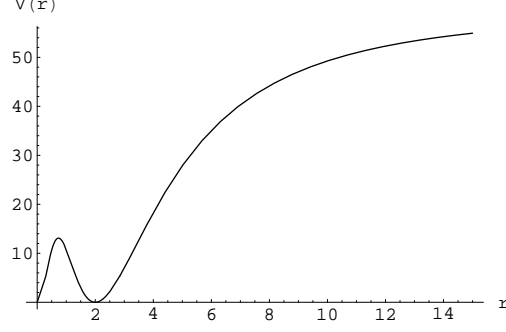


Figure 5: When $j_1 > j_{crit}$ there is a local potential minimum. Above, $j_1 = 10$ and $j_{crit} = 2$.

5.1 Scattering in the Plane $\theta = \frac{\pi}{2}$

The simplest motion of the supertube is that confined to a constant value of θ . The only trajectories of constant θ allowed by the equations of motion are for $\theta = \pi/2$ [2]. In this plane $\dot{\theta}$ and $\cos^2 \theta$ vanish and ϕ_2 is undefined, so the motion is purely radial and ΔE simplifies to a term containing \dot{r}^2 and an effective potential

$$\begin{aligned} V(r) &= \Delta E|_{\dot{r}=0, \dot{\theta}=0, \theta=\frac{\pi}{2}} \\ &= \tau_{D6} V_6 k \frac{F_{\sigma z} r^2 [j_1 / (\tau_{D6} V_6) - (H_{D0} + B_0^2 H_{D4}) r^2]^2}{2r^2 (H_{D0} + B_0^2 H_{D4}) (F_{\sigma z}^2 H_{F1} r^2 + H_{D0} H_{D4} r^4 + 2\omega F_{\sigma z})}. \end{aligned} \quad (5.13)$$

This effective potential approaches the limit

$$V_\infty = \frac{\tau_{D6} V_6 k F_{\sigma z} (1 + B_0^2)}{2} = \frac{1}{2} (\tau_{D0} \mathfrak{q}_{D0} + V_{T^4} \tau_{D4} \mathfrak{q}_{D4}) \quad (5.14)$$

at large r and vanishes as $r \rightarrow 0$. The potential also goes to zero, and a local minimum is created, when the expression in brackets in (5.13) vanishes. The latter occurs when

$$(H_{D0} + B_0^2 H_{D4}) r^2 = Q_{D0} + r^2 + B_0^2 (Q_{D4} + r^2) = \frac{j_1}{\tau_{D6} V_6} \quad (5.15)$$

$$\rightarrow \tau_{D6} V_6 (1 + B_0^2) r^2 = j_1 - j_{crit}, \quad (5.16)$$

keeping in mind that $j_{crit} = \tau_{D6} V_6 (Q_{D0} + B_0^2 Q_{D4}) = N_{D0} + B_0^2 \frac{\ell^4}{\alpha'^2} N_{D4}$.

Thus there is a stable minimum with $V = 0$ when

$$j_1 > j_{crit}, \quad (5.17)$$

at some $r_1 > 0$ where

$$r_1^2 = \frac{j_1 - j_{crit}}{\tau_{D6} V_6 (1 + B_0^2)}; \quad (5.18)$$

there are no minima for which $V \neq 0$. When $j_1 \leq j_{crit}$ the potential is attractive for all r , with no impediment to merging.

As noted in Section 4, in an adiabatic merger, supertubes with $j_1 > j_{crit}$ cannot merge with the black hole. Non-adiabatic mergers differ crucially in that such tubes only encounter a finite potential barrier, and thus a merger is possible even when $j_1 > j_{crit}$. Further inspection reveals that (5.15) is just (4.7), and (5.18) is (4.8), which gives the location of a BPS supertube, for $\theta = \pi/2$. This is consistent with the fact that a motionless supertube with $V = 0$ saturates the BPS bound and is thus in a BPS configuration.

5.2 Scattering for $\theta < \frac{\pi}{2}$

When $\theta < \pi/2$, we use (5.12) to arrive at the effective potential

$$\begin{aligned} V(r, \theta) &= \Delta E|_{\dot{r}=0, \dot{\theta}=0} \\ &= \tau_{D6} V_6 k \frac{F_{\sigma z} r^2}{2r^2 (H_{D0} + B_0^2 H_{D4}) (F_{\sigma z}^2 H_{F1} r^2 + H_{D0} H_{D4} r^4 \sin^2 \theta + 2\omega F_{\sigma z} \sin^2 \theta)} \\ &\quad \times \left(\frac{[j_1 / (\tau_{D6} V_6) - (H_{D0} + B_0^2 H_{D4}) r^2 \sin^2 \theta]^2}{\sin^2 \theta} + \frac{[j_2 / (\tau_{D6} V_6) - (Q_{D0} + B_0^2 Q_{D4}) \cos^2 \theta]^2}{\cos^2 \theta} \right). \end{aligned} \quad (5.19)$$

As with $V(r)$ in Section 5.1, $V(r, \theta)$ vanishes at $r = 0$ and approaches the value V_∞ at large r , independent of θ . The first term in the brackets of (5.19) always vanishes for some (r, θ) , while the second term vanishes when $j_2 \leq j_{crit}$, at the value θ_1 where

$$\cos \theta_1 = \sqrt{\frac{j_2}{j_{crit}}}; \quad (5.20)$$

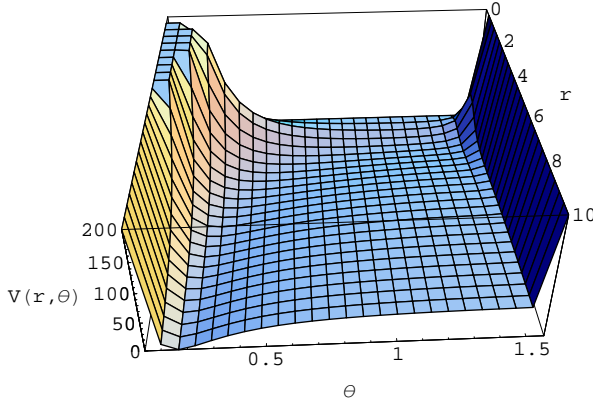


Figure 6: $V(r, \theta)$ for $j_2 \neq 0$ with $\tau_{D6} = V_6 = Q_{D4} = Q_{F1} = Q_{D0} = B_0 = j_2 = 1$, $k = 20$, $j_1 = 5$, $F_{\sigma z} = 3$, $\omega = \frac{1}{2}$.

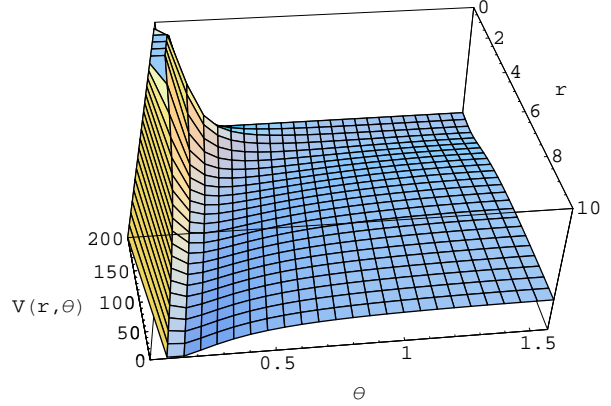


Figure 7: $V(r, \theta)$ with the same constants as in the previous figure, except that $j_2 = 0$. There is no potential wall at $\theta = \pi/2$.

equation (5.4) tells us that the points for which $\theta = \theta_1$ are characterized by vanishing $\dot{\phi}_2$. Clearly, for $V(r, \theta)$ to vanish when $r \neq 0$, both bracketed quantities of (5.19) must vanish at the same location. This is possible when

$$j_{crit} - j_1 < j_2 < j_{crit}, \quad (5.21)$$

in which case the vanishing location at (r_1, θ_1) constitutes a local minimum. We have r_1 given by

$$\tau_{D6} V_6 (1 + B_0^2) r_1^2 = \frac{(j_1 + j_2 - j_{crit}) j_{crit}}{j_{crit} - j_2} = j_{crit} \frac{\frac{j_1}{j_{crit}} - (1 - \cos^2 \theta_1)}{1 - \cos^2 \theta_1}. \quad (5.22)$$

Since a motionless supertube with $V = 0$ is BPS, it is natural that the conditions for the local minimum mirror the relations found in earlier sections for the BPS supertube. Indeed, (5.21) is (4.10), (5.22) is (4.9), and the vanishing of the first and second bracketed terms of $V(r, \theta)$ corresponds to the conditions (4.7) and (3.24) respectively.

Turning our attention to the dynamics of a non-BPS supertube, (5.4) shows that when $\theta = \pi/2$, $j_2 = 0$. Therefore, conservation of j_2 implies that when $j_2 \neq 0$ the supertube cannot reach $\theta = \pi/2$, a fact which manifests in a potential barrier there, in addition to the

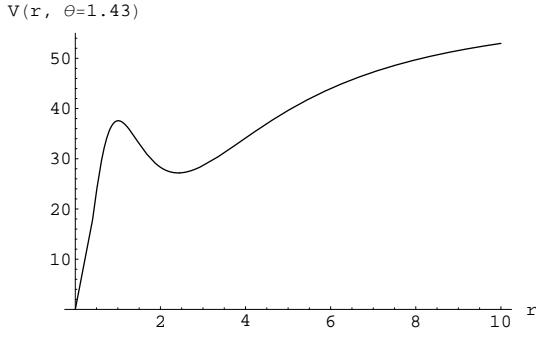


Figure 8: When $\theta = 1.43$, there is a peak and trough in r (but no corresponding behavior in θ). Here (and in the next figure) $\tau_{D6} = V_6 = Q_{D4} = Q_{F1} = Q_{D0} = B_0 = 1$, $k = 20$, $j_1 = 3$, $j_2 = 1.6$, $j_{crit} = 2$, $F_{\sigma z} = 3$, $\omega = \frac{1}{2}$. Thus the $j_{crit} - j_1 < j_2 < j_{crit}$ criterion for existence of a true minimum (not shown) is satisfied in both figures.

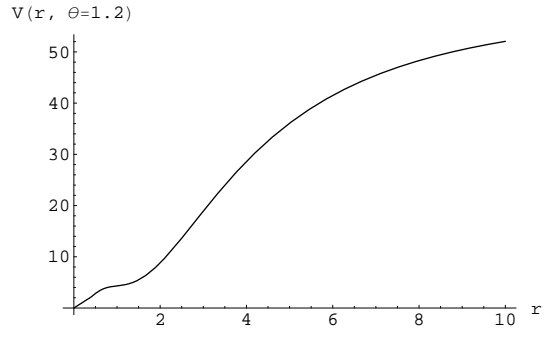


Figure 9: This is another cross-section of the same function $V(r, \theta)$ as the previous figure. Here, $\theta = 1.2$. For $r > 0$, $\partial_r V$ does not vanish, despite the fact that $j_{crit} - j_1 < j_2 < j_{crit}$ is satisfied.

one at $\theta = 0$. There are many points at which *either* $\partial_r V$ or $\partial_\theta V$ vanish, but the only true stationary point, where both vanish simultaneously, is the local minimum (r_1, θ_1) which exists when (5.21) is satisfied.

Equations (5.21, 5.22) reduce to those of Section 5.1 when $j_2 = 0$. However, the extrapolation from the $\theta = \pi/2$ case is not straightforward; the presence of the potential barrier is now a θ -dependent phenomenon. A potential barrier in the cross sections $V(r, \theta = \text{const.})$ appears for certain ranges of θ even when (5.21) is *not* satisfied.⁹ Figures 8 and 9 provide examples of cross sections with and without a potential barrier. The barrier is present for *all* θ when j_1 is sufficiently large, as it is for Figures 6 and 7. The precise conditions for this have proven elusive, as (5.21) is necessary but not sufficient.

We have seen that in the context of scattering analysis, BPS configurations are merely those of nonmoving supertubes at locations for which $V = 0$. Thus we conclude that with no external torque present, all BPS supertubes are in positions of stable equilibrium, separated from the black hole by a potential barrier. It is possible to overcome the barrier with

⁹Specifically, the barrier is present for all θ except the values $\theta_a < \theta < \theta_b$, where θ_a and θ_b are complicated functions of the parameters.

sufficient kinetic energy, or with the application of additional forces to the supertube; it can also quantum mechanically tunnel through the barrier. If we do introduce an appropriate external torque on the supertube in the ϕ_2 direction, it can move adiabatically from one BPS configuration another, as shown in Section 4. In the limit of vanishing velocity, the effective potential also vanishes. Our conclusion is that all BPS supertubes (for which the DBI approximation holds) can merge with the black hole by surmounting the potential barrier, while the subset with $j_1 \leq j_{crit}$ can also merge adiabatically.

For completeness we mention that the analogous condition to (5.21) for the D0-F1 supertube is

$$N_{D4} - j_1 < j_2 < N_{D4}. \quad (5.23)$$

6. Attempting to Violate the BMPV Bound

6.1 Results of the DBI Formalism

The BMPV black hole satisfies $J_1 = J_2 = J$ where J_i is its angular momentum in the plane of ϕ_i ; moreover, J is constrained by

$$J^2 < N_{D0} N_{D4} N_{F1}. \quad (6.1)$$

We consider a black hole whose angular momentum is very near this critical value. We keep $J_1 = J_2$ by letting two identical tubes with circumferential angular momenta $j = \mathfrak{q}_{F1}\mathfrak{q}_{D4}/k \ll J$ fall into it, one whose circumference is in the ϕ_1 direction and the other in the ϕ_2 direction. We limit ourselves to the case in which the motion of the tubes is confined to a single plane; $\theta = \pi/2$ for the ϕ_1 tube and $\theta = 0$ for the ϕ_2 tube. Thus these tubes carry angular momentum *only* along their circumferences. Thus the change in black hole angular momentum is

$$\Delta J = j = \frac{\mathfrak{q}_{F1}\mathfrak{q}_{D4}}{k}, \quad (6.2)$$

but for generality, we will keep ΔJ unspecified in our equations a while longer.

We expand $(J + \Delta J)^2$ with $\Delta N_{D0} = 2\mathfrak{q}_{D0}$, $\Delta N_{F1} = 2\mathfrak{q}_{F1}$ and $\Delta N_{D4} = 2\mathfrak{q}_{D4}$ and use the facts that

$$J \approx \sqrt{N_{D0}N_{F1}N_{D4}}, \quad (6.3)$$

$$\Delta J \ll J, \quad \mathfrak{q}_{D0} \ll N_{D0}, \quad \mathfrak{q}_{F1} \ll N_{F1}, \quad \mathfrak{q}_{D4} \ll N_{D4}. \quad (6.4)$$

Straightforward algebra reveals that if the two mergers discussed above are adiabatic, it is possible (naively at least) to produce a black object with $(J + \Delta J)^2 > (N_{D0} + 2\mathfrak{q}_{D0})(N_{F1} + 2\mathfrak{q}_{F1})(N_{D4} + 2\mathfrak{q}_{D4})$ if

$$N_{D0} < N^{spin} \equiv \frac{N_{F1}N_{D4}}{2(\mathfrak{q}_{F1}N_{D4} + \mathfrak{q}_{D4}N_{F1})^2} \times \left(\Delta J^2 - 2\mathfrak{q}_{D0}(\mathfrak{q}_{F1}N_{D4} + \mathfrak{q}_{D4}N_{F1}) + \Delta J \sqrt{\Delta J^2 - 4\mathfrak{q}_{D0}(N_{D4}\mathfrak{q}_{F1} + N_{F1}\mathfrak{q}_{D4})} \right); \quad (6.5)$$

that is, we can violate the BMPV bound (6.1) if $N_{D0} < N^{spin}$. For N^{spin} to be real the quantity in the square root must of course be nonnegative; it turns out that violation of the bound requires that it be positive. So in addition to (6.3-6.5) we require that

$$\Delta J^2 - 4\mathfrak{q}_{D0}(N_{D4}\mathfrak{q}_{F1} + N_{F1}\mathfrak{q}_{D4}) > 0 \quad (6.6)$$

for the BMPV bound to be exceeded.

We now rewrite the relation $j_1 > j_{crit}$ as

$$N_{D0} < \frac{\mathfrak{q}_{F1}\mathfrak{q}_{D4}}{k^2} - \frac{\mathfrak{q}_{D0}}{\mathfrak{q}_{D4}}N_{D4} \equiv N^{hump}. \quad (6.7)$$

We recall that the condition for allowing unobstructed mergers, $j_1 \leq j_{crit}$, is the opposite of this. Thus when $N_{D0} < N^{hump}$ there is no merger in the adiabatic case, and there is a hump in the effective potential in the non-adiabatic case. Now, a supertube, merging adiabatically or non-adiabatically, that satisfies (6.5) and (6.6) would cause a violation of the angular

momentum bound, *i.e.* “overspin” the black hole.¹⁰ An intriguing question naturally arises: must such a supertube satisfy (6.7)? Or is it possible to have $N^{spin} > N^{hump}$?

We now implement (6.2) and set $\Delta J = \mathbf{q}_{F1}\mathbf{q}_{D4}/k$. It happens that for given $\{\mathbf{q}_{F1}, \mathbf{q}_{D4}, k\}$ the maximum of N^{spin} is such that

$$N^{spin} \leq \frac{\mathbf{q}_{F1}\mathbf{q}_{D4}}{4k^2}, \quad (6.8)$$

and from (6.6), N_{D4} must satisfy

$$N_{D4} < \frac{\mathbf{q}_{F1}\mathbf{q}_{D4}^2}{4k^2\mathbf{q}_{D0}} - \frac{N_{F1}\mathbf{q}_{D4}}{4\mathbf{q}_{F1}}. \quad (6.9)$$

Combining (6.9) with (6.7) and (6.8) shows that for a supertube with enough angular momentum to overspin the black hole, the following will hold:

$$N^{hump} > \frac{3}{4} \frac{\mathbf{q}_{F1}\mathbf{q}_{D4}}{k^2} + N_{F1} \frac{\mathbf{q}_{D0}}{\mathbf{q}_{F1}} > N^{spin}. \quad (6.10)$$

So $N^{hump} > N^{spin}$ after all: as with the D0-F1 tube, it *is* possible to violate the BMPV bound, but such mergers must be non-adiabatic and produce a potential barrier. We have assumed the non-adiabatic mergers involve no perceptible change in Equations (6.5, 6.6).

In a non-adiabatic merger the presence of the energy ΔE , even if it is very small, implies that the final state of the merger is not BPS and thus not a BMPV black hole. Reference [2] discussed the process of using external forces to lift the supertube(s) over the potential barrier, and slowly lower it down the other side so that it merges with the black hole with an arbitrarily small ΔE . This would allow a violation of the angular momentum constraint on a non-BPS rotating black hole as well. Therefore it was suggested that the resulting object was not a black hole; rather, the black hole would fission into several black objects. The important issue of the nature of this final state is not addressed further in this note.

¹⁰ “Overspin” here merely denotes exceeding the BMPV bound; we are not suggesting that naked CTCs are created because, as mentioned below, we know the final product of the merger is not a BMPV black hole.

6.2 Comparison with the Supergravity Supertube

The issue of adiabatic mergers of black rings with BMPV black holes was treated in [11] and elaborated upon in [12]. The authors, using the same embedding as presented here, found exact BPS supergravity solutions for the black ring/black hole system, which in appropriate limits becomes a supertube/black hole system. Their findings allow for a comparison of the behavior during an adiabatic merger of the supergravity supertube with that found above for our worldvolume description. The supertube dipole charges will be given by $\{n^1, n^2, n^3\} = \{n_{D6}, n_{NS5} = 0, n_{D2}\}$, and the black hole charges by $\{N_1, N_2, N_3\} = \{N_{D0}, N_{F1}, N_{D4}\}$.

Some context for later results is provided by the supergravity description of the supertube *without* a black hole present. Contributions to the angular momenta of the supergravity supertube spacetime take the forms

$$j_{\Delta} = \tau_{D6} V_6 r^2 \sin^2 \theta (n_{D6} + \frac{\alpha'^2}{\ell^4} n_{D2}) \quad (6.11)$$

$$= \tau_{D6} V_6 r^2 \sin^2 \theta (1 + B_0^2),$$

$$j_{\xi} = \frac{1}{2} n^I \xi_I, \quad (6.12)$$

$$j_c = -\frac{1}{6} C_{IJK} n^I n^J n^K, \quad (6.13)$$

where $C_{IJK} = |\epsilon_{IJK}|$ and ξ_I is the charge of the supergravity supertube. The familiar j_1 and q_I from the worldvolume description correspond to the ‘microscopic’ angular momentum and charge of the supergravity supertube [8]. Microscopic quantities are those localized to the supertube itself, as opposed to those of the full spacetime. Here, the microscopic angular momentum is j_{Δ} , and the microscopic charges are given by

$$\bar{\xi}_I = \xi_I - \frac{1}{2} C_{IJK} n^J n^K \equiv \xi_I - \xi_c. \quad (6.14)$$

The asymptotic charges of the spacetime we will denote by \mathcal{N}_I , and the total spacetime

angular momenta by \mathcal{J}_1 and \mathcal{J}_2 . These are given by

$$\mathcal{J}_1 = j_\Delta + j_\xi + j_c, \quad (6.15)$$

$$\mathcal{J}_2 = j_\xi + j_c, \quad (6.16)$$

$$\mathcal{N}_I = \xi_I = \bar{\xi}_I + \xi_c. \quad (6.17)$$

The supergravity solution has a contribution to \mathcal{J}_2 arising from the supertube itself, given by $j_\xi + j_c$. The terms j_ξ , j_c and ξ_c are flux terms, meaning they arise from the R-R fields of the supertube. They are captured by flux integrals over the entire spacetime, but not by flux integrals taken near the ring itself, and thus it is unsurprising that they do not arise in the DBI results. Of course, since there are only two nonzero dipole charges, j_c vanishes.

We make the correspondence between the microscopic and worldvolume quantities,

$$j_1 \leftrightarrow j_\Delta, \quad q_I \leftrightarrow \bar{\xi}_I, \quad (6.18)$$

and observe that the replacement of localized charge of the supertube with full spacetime charge,

$$\bar{\xi}_I \rightarrow \xi_I, \quad (6.19)$$

reproduces \mathcal{J}_1 and \mathcal{N}_I of the supergravity solution (not \mathcal{J}_2 , however). In the worldvolume picture, (6.19) amounts to

$$q_{F1} \rightarrow q_{F1} + n_{D2} n_{D6}, \quad \text{or in general} \quad \mathbf{q}_{F1} \rightarrow \mathbf{q}_{F1} + \mathbf{n}_{D2} \mathbf{n}_{D6}. \quad (6.20)$$

We are now in a position to examine the changes that occur when the black hole is present. One difference is that a term

$$j_N = n^I N_I \sin^2 \theta \quad (6.21)$$

contributes to both angular momenta, and the microscopic angular momentum along the circumference is now $j_T = j_\Delta + j_N$. Now, in the supergravity picture we expect *a priori*

only that \mathcal{N}_I , \mathcal{J}_1 and \mathcal{J}_2 to be conserved quantities, but the adiabatic supergravity merger considered in [11] involves holding $N_I, \bar{\xi}_I, n^I, j_T$ and \mathcal{J}_1 constant.¹¹ This process requires an external torque in the ϕ_2 direction, and thus \mathcal{J}_2 varies. A significant result is that in the supergravity merger J_1^{final} and J_2^{final} are equal even after the addition of *one* supertube; the external torque must add the precise amount of ϕ_2 -momentum for this to hold. In fact, the final state of such a merger is merely another BMPV black hole, so there is no danger of violating the BMPV bound.

Other attributes of the supergravity solution are that for a merger to even take place,

$$j_T \leq n^I N_I \quad (6.22)$$

must be satisfied, and during the merger we have

$$\mathcal{J}_1 = j_T + j_\xi + J, \quad (6.23)$$

$$\mathcal{J}_2 = j_\xi + j_N + J, \quad (6.24)$$

$$\mathcal{N}_I = N_I + \bar{\xi}_I + \xi_c, \quad (6.25)$$

where J is again just the BMPV angular momentum.

We now compare the supergravity and worldvolume quantities. The quantity $n^I N_I$ is none other than j_{crit} . Moreover, examining (4.7) and (6.11) confirms that we can equate j_1 to the microscopic supergravity quantity $j_T = j_\Delta + j_N$; after the replacement (6.19) we also obtain j_ξ and ξ_c , thus fully accounting for \mathcal{J}_1 and \mathcal{N}_I . On the other hand, the worldvolume BPS value of j_2 , namely $j_{crit} \cos^2 \theta$, is not the same as its corresponding quantity $j_N = j_{crit} \sin^2 \theta$, so the DBI analysis fares no better than before with respect to accounting for \mathcal{J}_2 . Most importantly, the merger condition (4.16) corresponds exactly to (6.22). Overall, the level of agreement we have found seems encouraging, but there is certainly more to be understood.

¹¹Note that what the authors of [11] call the “embedding radius” is for us the coordinate $\rho_1 = r \sin \theta$ of the supertube. Also, what they call α for us is $\cot \theta$, so their $n^I N_I^{BH}/(1 + \alpha^2)$ is for us $n^I N_I \sin^2 \theta$.

7. Conclusions

The usefulness of the DBI description is that it allows us to treat time-dependent phenomena, and the success of this approach for the two-charge supertube motivated the present study. It is perhaps unsurprising that the basic attributes of the scattering process for three-charge supertube – the existence of a critical angular momentum, the existence of a stable minimum under certain conditions, the necessity of a potential barrier for any violation of the BMPV bound to take place – are very similar to those for the two-charge supertube. In addition, it is satisfying that in the limits outlined in Appendix B, our results reproduce those of the D0-F1 supertube.

Differences with the two-charge case can be seen in the findings of Section 6, some of which depend on the number k of constituent branes. Of great consequence is that the critical value of the angular momentum j_{crit} contains a term $\frac{\ell^4}{\alpha'^2} B_0^2 N_{D4}$. The existence of the magnetic field B_0 on the compact four-torus affects the dynamics of the supertube probe, including whether or not it is classically feasible for it to merge with the black hole. This term disappears for $\ell \ll \sqrt{\alpha'}$, which is consistent with the fact that our DBI methods do not apply in this regime.

Meanwhile, we have established a rough correspondence with the supergravity results of [11]. Such a comparison could of course be pursued further, in hopes of understanding the origin of those discrepancies and the physics underlying the replacement (6.19). It should be kept in mind that at the time of writing, the various worldvolume formulations of the three-charge supertube from [1] and [6] are still distinct. Reference [6] presented a worldvolume description of a calibrated supertube, and it would be interesting to see if that approach could lead to closer agreement with the supergravity solutions than was demonstrated here. And of course, a fair amount could be clarified about D0-D4-F1 (and

thus D1-D5-P) microstates, in the process of showing explicitly in any of these formulations that the supertube indeed has a discrete spectrum, in the manner of [13].

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A. The General Second-Order Lagrangian

The action S satisfies

$$S = \int \mathcal{L} d^7x = \int (\mathcal{L}_{DBI} + \mathcal{L}_{WZ}) d^7x \quad (\text{A.1})$$

where \mathcal{L} is the Lagrangian density. Our approximation involves the expansion of \mathcal{L} to second order in the velocities $\partial_t X^i$ and the gauge fields $F_{t\sigma}$ and $\delta F_{tz} = F_{tz} - 1$, using the fields given in (2.1-2.5) and following the methods outlined in the Appendix of [2]. In the absence of a specific embedding choice, \mathcal{L}_{WZ} is given by

$$\begin{aligned} (\tau_{D6} k)^{-1} \mathcal{L}_{WZ} = & \left(H_{D4}^{-1}(1 + \gamma_t) - 1 + B_0^2 [H_{D0}^{-1}(1 + \gamma_t) - 1] \right) F_{\sigma z} - (B_0^2 H_{D0}^{-1} + H_{D4}^{-1}) \gamma_\sigma \delta F_{tz} \\ & - (Q_{D0} + B_0^2 Q_{D4}) \psi_{t\sigma} \cos^2 \theta, \end{aligned} \quad (\text{A.2})$$

$$\text{where } \psi_{t\sigma} = \frac{\partial \phi_1}{\partial t} \frac{\partial \phi_2}{\partial \sigma} - \frac{\partial \phi_1}{\partial \sigma} \frac{\partial \phi_2}{\partial t}.$$

The full Lagrangian \mathcal{L} is

$$\begin{aligned}
(\tau_{D6} k)^{-1} \mathcal{L} = & -F_{\sigma z}(1 + B_0^2) - \left(Q_{D0} + B_0^2 Q_{D4} \right) \psi_{t\sigma} \cos^2 \theta \\
& + \frac{F_{\sigma z}}{2} \left(H_{D0} H_{F1} \Delta_{tt} + B_0^2 (H_{F1} H_{D4} \Delta_{tt}) \right) + \frac{F_{\sigma z}^{-3}}{2} H_{D0} H_{D4} (H_{D0} + B_0^2 H_{D4}) \Delta_{\sigma\sigma}^2 \delta F_{tz}^2 \\
& - F_{\sigma z}^{-2} (H_{D0} + B_0^2 H_{D4}) (H_{D0} H_{D4} \Delta_{\sigma t} - \gamma_\sigma \delta F_{tz}) \Delta_{\sigma\sigma} \delta F_{tz} \\
& - (H_{D0} + B_0^2 H_{D4}) (\Delta_{\sigma t} - \gamma_\sigma \Delta_{tt} + H_{F1} \Delta_{\sigma t} \delta F_{tz}) \\
& + \frac{F_{\sigma z}^{-1}}{2} (H_{D0} + B_0^2 H_{D4}) \left(F_{tz}^2 + H_{D0} H_{D4} \Delta_{\sigma\sigma} \Delta_{tt} + \delta F_{tz} (2\Delta_{\sigma\sigma} - 4\gamma_\sigma \Delta_{\sigma t} \right. \\
& \left. + H_{F1} \Delta_{\sigma\sigma} \delta F_{tz}) \right), \tag{A.3}
\end{aligned}$$

where, for $\xi = \{\sigma, t\}$, we have

$$\gamma_\xi = \gamma_1 \frac{\partial \phi_1}{\partial \xi} + \gamma_2 \frac{\partial \phi_2}{\partial \xi} \quad \text{and} \tag{A.4}$$

$$\Delta_{\xi\eta} = \frac{\partial r}{\partial \xi} \frac{\partial r}{\partial \eta} + r^2 \frac{\partial \theta}{\partial \xi} \frac{\partial \theta}{\partial \eta} + r^2 \sin^2 \theta \frac{\partial \phi_1}{\partial \xi} \frac{\partial \phi_1}{\partial \eta} + r^2 \cos^2 \theta \frac{\partial \phi_2}{\partial \xi} \frac{\partial \phi_2}{\partial \eta}. \tag{A.5}$$

After choosing the embedding $\phi_1 = \sigma$, these reduce to

$$\Delta_{\sigma\sigma} = r^2 \sin^2 \theta, \quad \Delta_{\sigma t} = 0, \quad \Delta_{tt} = \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \cos^2 \theta \dot{\phi}_2^2, \tag{A.6}$$

$$\gamma_\sigma = \gamma_1, \quad \gamma_t = \gamma_2 \dot{\phi}_2, \quad \psi_{t\sigma} = -\dot{\phi}_2. \tag{A.7}$$

We then arrive at

$$\begin{aligned}
(\tau_{D6} k)^{-1} \mathcal{L}_{WZ} = & \left(H_{D4}^{-1} (1 + \gamma_2 \dot{\phi}_2) - 1 + B_0^2 [H_{D0}^{-1} (1 + \gamma_2 \dot{\phi}_2) - 1] \right) F_{\sigma z} \\
& - (B_0^2 H_{D0}^{-1} + H_{D4}^{-1}) \gamma_1 \delta F_{tz} + (Q_{D0} + B_0^2 Q_{D4}) \dot{\phi}_2 \cos^2 \theta \tag{A.8}
\end{aligned}$$

for \mathcal{L}_{WZ} , and (5.1) for \mathcal{L} .

B. Relations Between the D0-D4-F1 Supertube and the D0-F1 Supertube

The corresponding formulae for the D0-F1 supertube can be obtained from those presented

above for the D0-D4-F1 supertube by performing the following substitutions:

$$B_0 \rightarrow 0, \quad (\text{B.1})$$

$$k \rightarrow 1, \quad (\text{B.2})$$

$$\frac{\ell^4}{\alpha'^2} \rightarrow \frac{\alpha'^2}{\ell^4}, \quad (\text{B.3})$$

$$\{N_{D0}, N_{F1}, N_{D4}\} \rightarrow \{N_{D4}, N_{F1}, N_{D0}\}, \quad (\text{B.4})$$

$$\{q_{D0}, q_{F1}, q_{D4}\} \rightarrow \{0, q_{F1}, q_{D0}\}, \quad (\text{B.5})$$

$$\{n_{D2}, n_{D6}\} \rightarrow \{0, n_{D2}\}, \quad (\text{B.6})$$

$$\{\tau_{D0}, \tau_{F1}, \tau_{D4}\} \rightarrow \{\tau_{D4}, \tau_{F1}, \tau_{D0}\}, \quad (\text{B.7})$$

$$\{\tau_{D6}, V_6\} \rightarrow \{\tau_{D2}, V_2\}. \quad (\text{B.8})$$

This process represents several steps. First, the magnetic field B_0 on the compact torus is sent to zero, eliminating any D0 charge or D2 dipole charge:

$$\{q_{D0}, q_{F1}, q_{D4}\} \rightarrow \{0, q_{F1}, q_{D4}\}, \quad (\text{B.9})$$

$$\{n_{D2}, n_{D6}\} \rightarrow \{0, n_{D6}\}. \quad (\text{B.10})$$

Next the number of coincident D6 branes, k , is set to one. This leaves us with a D6 brane D4-F1 supertube, T-dual to the D2 brane D0-F1 supertube. Thus the following step is a T-duality transformation performed on each of the compact directions $\{x^6, x^7, x^8, x^9\}$, which effects the change (B.3). The D4 branes wrapped on the T^4 direction become D0 branes and vice versa, leading to (B.4). At the same time the single D6 brane of the D0-D4-F1 supertube becomes the D2 brane of the D0-F1 supertube (requiring the replacements (B.7) and (B.8)), and so we have

$$\{0, q_{F1}, q_{D4}\} \rightarrow \{0, q_{F1}, q_{D0}\}, \quad (\text{B.11})$$

$$\{0, n_{D6}\} \rightarrow \{0, n_{D2}\}. \quad (\text{B.12})$$

yielding (B.5-B.6).

Equivalent to the substitution $V_6 \rightarrow V_2$ is $V_{T^4} \rightarrow 1$, which facilitates determination of the D0-F1 counterparts to equations such as (3.26) and (5.11). We also note that the substitutions for the quantities Q_{D0} and Q_{D4} necessarily involve τ_{D6} and V_6 . Thus

$$\tau_{D6} V_6 Q_{D0} \rightarrow \tau_{D2} V_2 Q_{D4} \quad i.e. \quad N_{D0} \rightarrow N_{D4}, \quad (\text{B.13})$$

$$\tau_{D6} V_6 Q_{D4} \rightarrow \tau_{D2} V_2 Q_{D0} \quad i.e. \quad \frac{\ell^4}{\alpha'^2} N_{D4} \rightarrow \frac{\alpha'^2}{\ell^4} N_{D0}. \quad (\text{B.14})$$

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